Theory of free fermions under random projective measurements

Igor Poboiko

Karlsruher Institut für Technologie

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In collaboration with: **Paul Pöpperl**, **Igor Gornyi**, **Alexander Mirlin**

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Outline

- Introduction: measurement-induced phase transitions, entanglement entropy and particle number cumulants
- Field theory approach: Keldysh & replica path integral, non-linear sigma-model
- Numerical evidence
- Extensions to $d > 1$
- Outlook

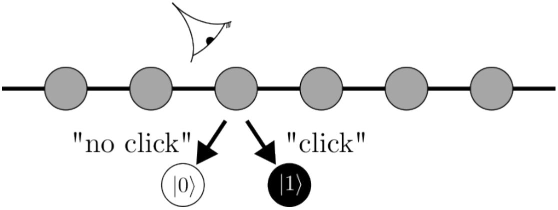
Introduction: what's that all about?

The model

- Unitary evolution of 1D Fermi gas:

$$\hat{H} = -J \sum_{\langle xx' \rangle} (\hat{\psi}_x^\dagger \hat{\psi}_{x'} + h.c.)$$

- **Random** (rate per site γ) local projective measurements:

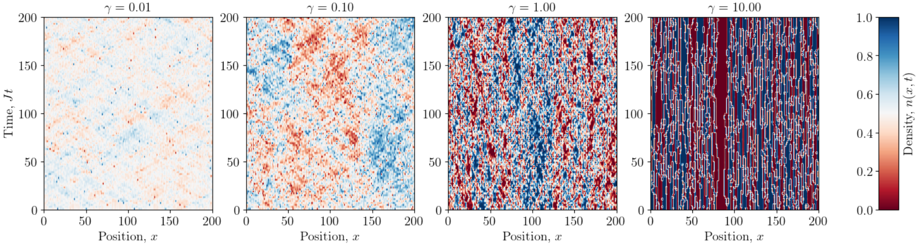


- Born rule:

$$\text{Prob}(n|(x,t)) = \langle \Psi(t) | \hat{\mathbb{P}}_n(x) | \Psi(t) \rangle$$

$$|\Psi(t+0)\rangle = \frac{\hat{\mathbb{P}}_n(x)\,|\Psi(t)\rangle}{\left\| \hat{\mathbb{P}}_n(x)\,|\Psi(t)\rangle \right\|}, \qquad \begin{cases} \hat{\mathbb{P}}_0(x) & = 1 - \hat{\psi}_x^\dagger \hat{\psi}_x \\ \hat{\mathbb{P}}_1(x) & = \hat{\psi}_x^\dagger \hat{\psi}_x \end{cases}$$

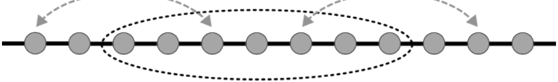
Density evolution



Qualitatively different wavefunctions:

- $\gamma \ll J$: weak fluctuations around $\langle \hat{n} \rangle \approx 0.5$, ballistic spreading
- $\gamma \gg J$: quantum Zeno limit, $\langle \hat{n} \rangle \in \{0, 1\}$, rare jumps due to unitary evolution

Quantitative characteristics



- Entanglement entropy:

$$\mathcal{S}_A = -\operatorname{Tr}(\hat{\rho}_A \ln \hat{\rho}_A), \quad \hat{\rho}_A(T) = \operatorname{Tr}_{\bar{A}} |\Psi\rangle \langle\Psi|$$

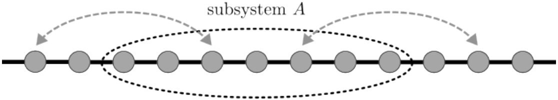
- For Gaussian states, related to full counting statistics $\hat{N}_A = \sum_{x\in A} \hat{\psi}_x^\dagger \hat{\psi}_x$:

$$\mathcal{S}_A \approx \frac{\pi^2}{3}\mathcal{C}_A^{(2)} + \frac{\pi^4}{45}\mathcal{C}_A^{(4)} + \ldots, \quad \mathcal{C}_A^{(N)} = \left\langle\left\langle \hat{N}_A^N \right\rangle\right\rangle$$

Klich, Levitov, 2009

- NB: nonlinear (in density matrix) quantities! E.g.

$$\overline{\operatorname{Tr}(\hat{\rho}\hat{N}_A)\operatorname{Tr}(\hat{\rho}\hat{N}_A)} \neq \operatorname{Tr}(\bar{\hat{\rho}}\hat{N}_A)\operatorname{Tr}(\bar{\hat{\rho}}\hat{N}_A)$$



Possible phases:

Volume law	**Area law**	**Intermediate**
$\mathcal{S}_A, \mathcal{C}_A^{(2)} \propto \|A\|$	$\mathcal{S}_A, \mathcal{C}_A^{(2)} \propto \|\partial A\| = \text{const}$	$\mathcal{S}_A, \mathcal{C}_A^{(2)} \propto \|\partial A\| \ln \|A\|$
Typical for "generic" Hilbert space state	Typical for low-T state (finite correlation length)	e.g. zero-T Fermi gas (power-law correlations)

Questions: what are the phases? Quantum phase transition?
(control parameter — rate of measurements γ/J)

Previous findings: interacting systems

- **Random unitary circuits**: Volume-law to area-law transition
  Numerics & analytical mapping on known statmech models
  Li, Chen, Fisher, 2018, 2019
  Chan, Nandkishore, Pretko, Smith, 2019
  Bao, Choi, Altman, 2020
- **Interacting many-body Hamiltonian systems**
  Numerical evidence for volume-law to area-law transition
  Tang, Zhu, 2020
  Goto, Danshita, 2020
  Fuji, Ashida, 2020
  Doggen, Gefen, Gornyi, Mirlin, Polyakov, 2022

(and a lot more)

Previous findings: free fermions

- **Numerics (+ phenomenology):**
  - transition between critical (log) and area-law phases:
    Alberton, Buchhold, Diehl, 2021
    Turkeshi, Biella, Fazio, Dalmonte, Schiro, 2021
    Minato, Sugimoto, Kuwahara, Saito, 2022
    Szyniszewski, Lunt, Pal, arXiv:2211.0253
  - area law: Cao, Tilloy, Luca, 2019
  - area law (with intermediate critical behavior):
    Coppola, Tirrito, Karaevski, Collura, 2022
- **Field theory**: Buchhold, Minoguchi, Altland, Diehl, 2021
  Effective Luttinger Liquid behavior with BKT-type transition between critical log-phase and area-law phase
  (but inconsistent with numerics)

Related models

Related models, but with other universality classes

- Non-unitary evolution (no measurements)
  Chen, Li, Fisher, Lucas, 2020: single critical log phase (CFT)
  Jian, Bauer, Keselmann, Ludwig, 2022: Majorana random circuits critical to area law transition
- Recent (parallel) work, similar to our approach, Majorana random circuits (w/ measurements)
  Jian, Shapourian, Bauer, Ludwig, arXiv:2302.09094
  Fava, Piroli, Swann, Bernard, Nahum, arXiv:2302.12820
  critical $\ln^2 l$ phase; transition to area-law

General approach: key points

Key object: non-normalized density matrix

- For a fixed "measurement trajectory" $\mathcal{T} = \{x_m, t_m, n_m\}_{m=1}^{M}$, we define **non-normalized density matrix** $\hat{D}$, which obeys **linear evolution**.
- Initial condition:

$$\hat{D}(0) = |\Psi_0\rangle \langle\Psi_0| = \hat{\rho}_0$$

- Between consecutive measurements $\Delta t = t_m - t_{m-1}$: unitary evolution:

$$\hat{D}(t_m) = \hat{U}_0(\Delta t)\hat{D}(t_{m-1})\hat{U}_0^\dagger(\Delta t)$$

- Measurement:

$$\hat{D}(t_m + 0) = \hat{\mathbb{P}}_{n_m}(x_m)\hat{D}(t_m - 0)\hat{\mathbb{P}}_{n_m}(x_m)$$

- Density matrix (NB: pure state!): $\hat{\rho} = \hat{D}/\operatorname{Tr}\hat{D}$
- Normalization: **Born rule**

$$\operatorname{Tr}\hat{D} = \operatorname{Prob}(\{n_m\}|\{x_m, t_m\})$$

Replica trick

Need to average non-linear in $\hat{\rho}$ objects (trajectory post-selection is required to <u>measure</u> it in experiment)

- Particle number cumulant:

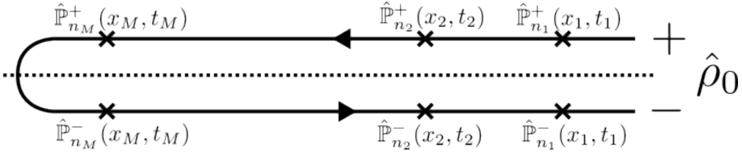
$$\mathcal{C}_A^{(N)} = \left\langle\left\langle \hat{N}_A^N \right\rangle\right\rangle = \cdots + (-1)^N \left\langle \hat{N}_A \right\rangle^N$$

- N copies of density matrix required:

$$\overline{\hat{\rho}^{\otimes N}}^{(x,t,n)} = \overline{\sum_{\{n_m\}} \mathrm{Tr}\, \hat{D} \cdot \frac{\hat{D}^{\otimes N}}{\mathrm{Tr}^N \hat{D}}}^{(x,t)} = \lim_{R\to 1} \mathrm{Tr}_{r=N+1,\ldots,R} \overline{\sum_{\{n_m\}} \hat{D}^{\otimes R}}^{(x,t)}$$

- For $N > 1$, analytic continuation from $R \geq N$ to $R \to 1$ is required (**replica trick**)
- Number of replicas of $\hat{D}$ is $R \to 1$, indepedent of N! (consequence of Born rule)

Keldysh action



$$\hat{D}(\mathcal{T}) = \mathcal{T}_{\mathcal{C}} \left\{ \hat{\rho}_0 \hat{U}_C \prod_{i=1}^{M} \hat{\mathbb{P}}^+_{n_i}(x_i, t_i) \hat{\mathbb{P}}^-_{n_i}(x_i, t_i) \right\}$$

path integral representation + average over “measurement trajectories”
$\longrightarrow$ replicated Keldysh action

$$\mathcal{L}[\bar{\psi}, \psi] = \sum_r \bar{\psi}_r (i\partial_t - \hat{H}_0)\psi_r + \gamma \mathcal{L}_M[\bar{\psi}, \psi]$$

$$i\mathcal{L}_M[\bar{\psi}, \psi] = \prod_{r=1}^{R} \bar{\psi}^+_r \psi^+_r \bar{\psi}^-_r \psi^-_r + \prod_{r=1}^{R} (1 - \bar{\psi}^+_r \psi^+_r)(1 - \bar{\psi}^-_r \psi^-_r) - 1$$

Local in space-time, $4R$ Fermionic interaction! γ — measurement rate

Density correlation functions

- "Replica symmetric" correlation function (fluctuations of **average** density):
$$C_0(x,t) = \overline{\langle \{\hat{n}(x,t), \hat{n}(0,0)\}\rangle /2} - n^2$$
- "Replica-offdiagonal" correlation function (of main interest):
$$C(x,t) = \overline{\langle \{\hat{n}(x,t), \hat{n}(0,0)\}/2\rangle} - \overline{\langle \hat{n}(x,t)\rangle \langle \hat{n}(0,0)\rangle}$$
- Can be obtained within replica trick:
$$C_{rr'}(x,t) = \langle\langle \delta n_r(x,t) \delta n_{r'}(0,0)\rangle\rangle = C_0(x,t) - C(x,t)(1-\delta_{rr'})$$
- Second cumulant:
$$\mathcal{C}_l^{(2)} = \langle\langle \delta N_l^2 \rangle\rangle = \int_0^l dx \int_0^l dy \; C(x-y, t=0)$$

Non-linear sigma-model
$(\gamma/J \ll 1)$

Non-linear sigma-model

- Generalized Hubbard-Stratanovich transformation
  $\longrightarrow$ matrix $(2R \times 2R)$ field theory:

$$Q_{ab}(x,t) \simeq \psi_a(x,t)\bar{\psi}_b(x,t)/2$$

- Manifold: $\hat{Q} = \hat{\mathcal{R}}^{-1}\hat{\Lambda}\hat{\mathcal{R}}, \quad \hat{Q}^2 = 1, \quad \mathrm{Tr}\,\hat{Q} = 0, \quad (U(2R)/U(R) \times U(R))$
- Lagrangian:

$$i\mathcal{L}[\hat{Q}] = \mathrm{tr}\left(\frac{1}{2}\hat{\Lambda}\hat{\mathcal{R}}^{-1}\partial_t\hat{\mathcal{R}} - \frac{D}{8}(\partial_x\hat{Q})^2\right) + i\gamma\mathcal{L}_M[\hat{Q}]$$

$$i\mathcal{L}_M[\hat{Q}] = \det\left(\frac{1-\hat{Q}\hat{\tau}_x}{2}\right) + \det\left(\frac{1+\hat{Q}\hat{\tau}_x}{2}\right) - 1$$

- Saddle point $\Leftrightarrow$ self-consistent Born approximation:

$$\hat{Q} = \hat{\Lambda} = \begin{pmatrix} 1 & 2(1-2n) \\ 0 & -1 \end{pmatrix}_K \otimes \hat{1}_R$$

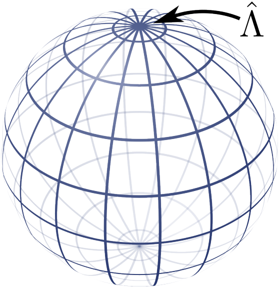
  - Retarded/advanced Green functions: $G_{R/A}(E,k) = (E + 2J\cos k \pm i\gamma)^{-1}$
  - Keldysh Green function: $G_K = (1-2n)(G_R - G_A)$
  - Infinite temperature heating!

NLSM: replica symmetric case $R = 1$

- Equivalent to properties of $\overline{\hat{\rho}(t)}$. We can directly set $R = 1 \Rightarrow \mathcal{L}_M \equiv 0$.
- Manifold: $U(2)/U(1) \times U(1) \simeq \mathcal{S}_2$
- Soft modes $\longrightarrow$ diffusion:

$$\mathcal{D}(\omega, q) = (Dq^2 - i\omega)^{-1},$$

- Inelastic mean free time $\tau_0 = 1/2\gamma$
- Root mean square velocity $v_0 = \int_{-\pi}^{\pi} v^2(k) = 2J^2$
- Mean free path $l_0 = v_0 \tau_0 = J/\gamma\sqrt{2}$
- Diffusion constant $D = v_0^2 \tau_0 = J^2/\gamma$



$\longrightarrow$ "replica symmetric" correlation function:

$$C_0(x,t) = \overline{\langle \{\hat{n}(x,t), \hat{n}(0,0)\} \rangle /2} - n^2 \simeq n(1-n) \frac{\exp\left(-x^2/4D|t|\right)}{\sqrt{4\pi D|t|}}$$

Renormalization is absent! (due to instantaneous "interaction")

NLSM: replicon sector

- Manifold: $U(2R)/U(R) \times U(R)$, $\quad \hat{Q} = \hat{\mathcal{R}}^{-1}\hat{\Lambda}\hat{\mathcal{R}}$.
  Can be parametrized as:

$$\hat{Q} = \hat{\mathcal{R}}_\Phi^{-1}\hat{\mathcal{R}}_\Theta^{-1}\hat{Q}_0\hat{\mathcal{R}}_\Theta\hat{\mathcal{R}}_\Phi, \quad \hat{\mathcal{R}}_\Theta = e^{-i\hat{\Theta}\hat{\tau}_y/2}, \quad \hat{\mathcal{R}}_\Phi = e^{-i\hat{\Phi}\hat{\tau}_x/2}$$

  $\hat{Q}_0$ — replica symmetric matrix
  $\hat{\Phi}, \hat{\Theta}$ — traceless $R \times R$ matrices

- "Measurement" term in action independent of $\hat{\Phi}$:

$$i\mathcal{L}_M[\hat{Q}] = \det\left(\frac{1-\hat{Q}\hat{\tau}_x}{2}\right) + \det\left(\frac{1+\hat{Q}\hat{\tau}_x}{2}\right) - 1 = i\mathcal{L}_M[\hat{\Theta}]/\gamma$$

  $\longrightarrow SU(R)$ massless mode $\hat{U} = \exp(i\hat{\Phi})$

- Integrate out Θ modes $\longrightarrow$ **emergent (d+1) NLSM**:

$$i\mathcal{L} = -\frac{g_0}{2}\,\mathrm{Tr}\left(v_0^{-1}\partial_t\hat{U}^{-1}\partial_t\hat{U} + v_0\partial_x\hat{U}^{-1}\partial_x\hat{U}\right), \quad g_0 = 2l_0 n(1-n) \gg 1,$$

NLSM: replicon physics (Gaussian fluctuations)

- Respect causality $\longrightarrow$ "absorbing boundary" at $t = t_f$, $\hat{U}(x, t_f) = \hat{\mathbb{I}}$.
- Equal time correlation function:

$$C(x - x', t = 0) \simeq -\frac{g_0^2}{v_0^2} \left\langle \partial_t \hat{\Phi}(x, t_f) \partial_t \hat{\Phi}(x', t_f) \right\rangle$$

- Gaussian level:

$$C(x) \simeq n(1-n)\delta(x) - \frac{g_0}{\pi x^2}, \quad C(q) = g_0|q|$$

Second cumulant:

$$\mathcal{C}_l^{(2)} = \int_0^l dx \int_0^l dy\, C(x-y) = \int_0^{\sim l_0^{-1}} \frac{dq}{\pi} \frac{C(q)}{q^2}(1 - \cos ql) \simeq \frac{2g_0}{\pi} \ln \frac{l}{l_0},$$

- **Critical (logarithmic) behaviour!** $\mathcal{C}_l^{(2)}, S_l \propto l_0 \ln \frac{l}{l_0}, \quad l_0 \propto \gamma^{-1} \gg 1$

Quantum fluctuations: "Anderson localization"

- 1D measurement problem $\longrightarrow$ 2D SU(R) NLSM

$$i\mathcal{L} = -\frac{g_0}{2}\,\mathrm{Tr}\left(\partial_\mu \hat{U}^{-1}\partial_\mu \hat{U}\right), \quad R \to 1$$

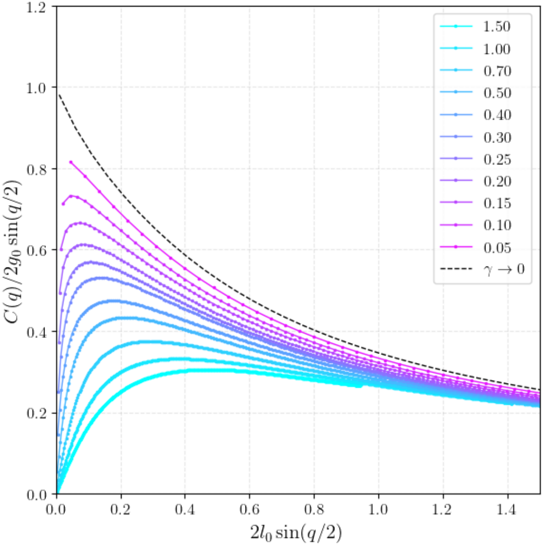
- Renormalization is known since ages (Hikami, Wegner, ... (1980s))
- RG β-function:

$$\frac{dg}{d\ln l} = \beta(g) = -\frac{R}{4\pi} + O(1/g)$$

- Theory flows to strong coupling (cf. 2D Anderson localization)
  Correlation ("localization") length $l_{\mathrm{corr}} \simeq l_0 \exp\left(4\pi g_0\right)$
- Exponential decay of correlators $C(x) \sim \exp(-|x|/l_{\mathrm{corr}})$
- **Crossover between log law at $l \lesssim l_{\mathrm{corr}}$ and area law at $l \gtrsim l_{\mathrm{corr}}$!**

Numerical evidence

Density correlation function in 1D

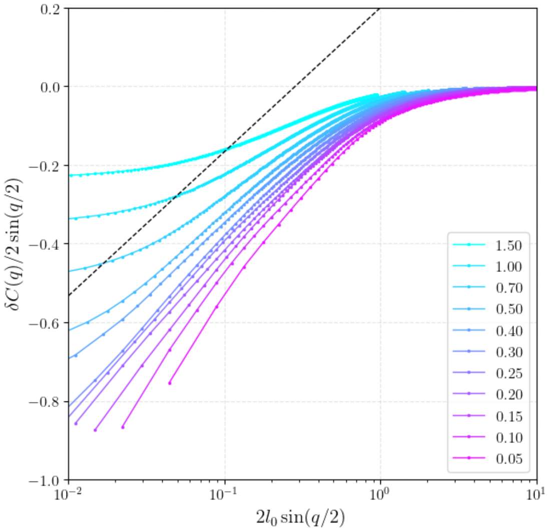


Behavior of $C(q)/q$:

- $1/|q| \Rightarrow$ Volume law
- const $\Rightarrow$ Critical (log) law (Gaussian result)
- $0 \Rightarrow$ area law ("Localization")

(dashed: ballistic analytic)

"Weak localization"



Precursors of localization:

$$C(q \to 0)/|q| \simeq g(q)$$

$$\approx g_0 - C \ln \frac{1}{q l_0}$$

(dashed line: $C = 1/2\pi$)

Extensions to $D > 1$

Above 1D: measurement-induced phase transition

- Measurement problem in dim $= d \quad \longrightarrow \quad SU(R)$ NLSM in dim $= d+1$
- Gaussian approximation: $C(q) \approx g_0|q|$
  Cumulant, entanglement entropy $\longrightarrow$ critical (log) law:

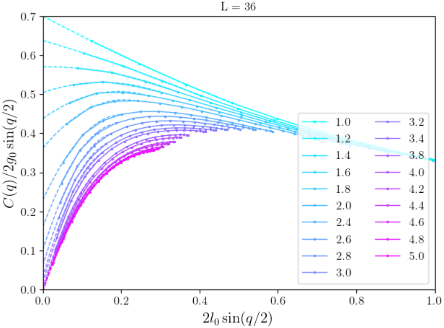
$$\mathcal{C}_l^{(2)}, S_l \propto l_0 l^{d-1} \ln(l/l_0)$$

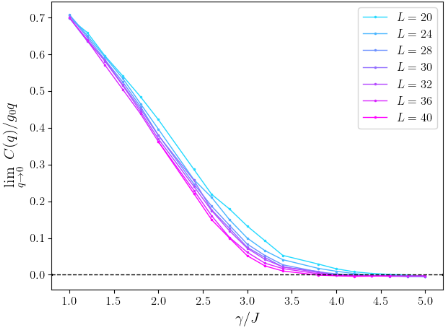
- RG β-function for "dimensionless conductance" $G = gl^{d-1}$:

$$\frac{dG}{d \ln l} = (d-1)G - \frac{R}{4\pi} + O(1/G)$$

- $d > 1 \longrightarrow$ phase transition at $\gamma = \gamma_c$
  $\gamma < \gamma_c$: critical law $\mathcal{C}_l^{(2)}, S_l \propto l^{d-1} \ln l$
  $\gamma > \gamma_c$: area law $\mathcal{C}_l^{(2)}, S_l \propto l^{d-1}$

Free fermions on square lattice $L \times L$





$\longrightarrow$ **transition at** $\gamma_c/J \approx 3.0$

Closing remarks

Outlook

- Weak / continuous measurements
  $\longrightarrow$ same universality class
- Non-unitary evolution (instead of measurements)
  $\longrightarrow$ similar NLSM with replica limit $R \to 0$
- Interactions (e.g. Hubbard model / XXZ spin chain)
  $\longrightarrow$ transition between volume and area law expected
- "Superconductivity" (e.g. anisotropic XY spin chain)
  $\longrightarrow$ other symmetry class instead of $SU(R)$
- Kinetics and purification
- Power-law hopping
- Measurement-only dynamics (measuring non-commuting observables)
- ...and a lot more!

Summary

- Keldysh technique + replica trick $\longrightarrow$ **Non-Linear Sigma-Model**
- Two sectors of NLsM:
  - Replica symmetric: diffusive $U(2)/U(1)\times U(1)$ NLSM
  - Replicon: $d+1$ (space+time) $SU(R)$ NLSM with $R\to 1$
- $d=1$: crossover between log- and area-law at $l_{\text{corr}}\sim l_0\exp(4\pi g_0)$

$$\mathcal{C}_l^{(2)}\simeq n(1-n)\cdot\begin{cases} l, & l\ll l_0\\ \frac{4}{\pi}l_0\ln\frac{l}{l_0}, & l_0\ll l\ll l_{\text{corr}}\\ \sim l_0^2, & l_{\text{corr}}\lesssim l\end{cases}$$

  with $l_0=J/\gamma\sqrt{2}\gg 1$ and $g_0=2l_0n(1-n)\gg 1$
  No transition, only area law phase!
- $d>1$: **phase transition between log- and area-law phases**
- Supported by **direct numerical simulations**

Thank you for your attention!